

# A Free $\mathcal{N} = 2$ Supersymmetric System: Novel Symmetries

S. Krishna<sup>(a)</sup>, R. P. Malik<sup>(a,b,c)</sup>

<sup>(a)</sup> *Physics Department, Centre of Advanced Studies,  
Banaras Hindu University (BHU), Varanasi - 221 005, India*

<sup>(b)</sup> *DST-CIMS, BHU, Varanasi - 221 005, (U.P.), India*

<sup>(c)</sup> *AS-ICTP, Strada Costiera-11, I-34014, Trieste, Italy*

E-mails: skrishna.bhu@gmail.com; rpmaalik1995@gmail.com

**Abstract:** We discuss a set of novel discrete symmetries of a free  $\mathcal{N} = 2$  supersymmetric (SUSY) quantum mechanical system which is the limiting case of a widely-studied interacting SUSY model of a charged particle constrained to move on a sphere in the background of a Dirac magnetic monopole. The *usual* continuous symmetries of this model provide the physical realization of the de Rham cohomological operators of differential geometry. The interplay between the *novel* discrete symmetries and *usual* continuous symmetries leads to the physical realization of relationship between the (co-)exterior derivatives of differential geometry. We have also exploited the supervariable approach to derive the nilpotent  $\mathcal{N} = 2$  SUSY symmetries of the theory and provided the geometrical origin and interpretation for the nilpotency property. Ultimately, our present study (based on innate symmetries) proves that our *free*  $\mathcal{N} = 2$  SUSY example is a tractable model for the Hodge theory.

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# 1 Introduction

It is a well-known fact that *three* out of *four* fundamental interactions of nature are theoretically described within the framework of gauge theories which are characterized by the existence of local gauge symmetries at the *classical* level (see, e.g. [1]). A very special class of gauge theories is endowed with the dual-gauge symmetries, too. For instance, it has been shown recently that any arbitrary Abelian  $p$ -form ( $p = 1, 2, 3, \dots$ ) gauge theory would be always endowed with the (dual-)gauge symmetries in  $D = 2p$  dimensions of spacetime [2,3,4]. As a consequence, such theories have been shown, within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism, to respect (at the *quantum* level) the (anti-)BRST and (anti-)dual-BRST symmetries in the Feynman gauge (see, e.g. [2-5] for details). Mathematically, such field theoretic models have been shown to present a set of tractable examples for the Hodge theory (see, e.g. [3-9]) because the continuous and discrete symmetries of such a class of theories provide the physical realizations of the de Rham cohomological operators of differential geometry.

In a recent set of papers (see, e.g. [10-12]), a collection of  $\mathcal{N} = 2$  SUSY quantum mechanical models (QMM) have *also* been shown to represent the models for the Hodge theory where the symmetries (and their generators) play an important role. The central purpose of our present endeavor is to show that the free  $\mathcal{N} = 2$  SUSY QMM (which is the limiting case of the physically interesting model of a charged particle, constrained to move on a sphere, in the background of a Dirac magnetic monopole) *also* presents an example for the Hodge theory. This is essential *first* modest step for us if we wish to prove, in a systematic manner, that its  $\mathcal{N} = 2$  and  $\mathcal{N} = 4$  *interacting* versions are also models for the Hodge theory.

There are a few continuous symmetries (and their conserved charges) and discrete symmetries that are urgently needed to prove a field theoretic model and/or a SUSY QMM to be an example for the Hodge theory. Such studies are physically relevant because, taking the help of this kind of investigations, we have proven that the  $(1 + 1)$ -dimensional (2D) (non-)Abelian gauge theories (without any interaction with the matter fields) are perfect examples of a *new* class of topological field theory (TFT) which capture a few key aspects of the Witten-type TFT and some of the salient features of Schwartz-type TFT (see, e.g. [13]). An interacting system of 2D photon and Dirac fields has also been shown to be a model for the Hodge theory where the topological gauge field couples with the matter fields [9,14]. Similar is the case with our very recent works on the modified versions of the 2D anomalous gauge theory and Proca theory where matter and gauge fields are present together [15,16].

The free  $\mathcal{N} = 2$  SUSY system under consideration is interesting in its own right because, since the seminal work by Dirac [17], the system of charged particle and magnetic monopole has been studied from different angles due to its rich mathematical and physical structures (see, e.g. [18,19]). The  $\mathcal{N} = 2$  (and its generalization to  $\mathcal{N} = 4$ ) superfield formulations have been carried out in [20,21] where the quantum mechanical Lagrangian for the above physical system has been obtained by adopting the  $CP^{(1)}$  model approach so that there is *no singularity* in the monopole interaction. In our present investigation, we take the *free*  $\mathcal{N} = 2$  SUSY version of the Lagrangian obtained in [20] and show that it provides the physical model for a Hodge theory due to its innate symmetries.

The material of our present paper is organized in the following fashion. In Sec. 2, we recapitulate the bare essentials of the  $\mathcal{N} = 2$  nilpotent ( $s_1^2 = s_2^2 = 0$ ) SUSY transformations ( $s_1$  and  $s_2$ ) and a bosonic symmetry  $s_\omega = \{s_1, s_2\}$ . Our Sec. 3 is devoted to the discussion of a set of discrete symmetry transformations. In Sec. 4, we lay emphasis on the algebraic structures of the symmetry transformations and corresponding conserved charges. The  $\mathcal{N} = 2$  SUSY continuous nilpotent symmetry transformations (i.e.  $s_1$  and  $s_2$ ) are derived by exploiting the SUSY invariant restrictions in Sec. 6. Finally, we make some concluding remarks in Sec. 7.

## 2 Preliminaries: usual continuous symmetries

We begin with the  $\mathcal{N} = 2$  SUSY invariant Lagrangian (derived by exploiting the standard technique of the superspace formalism) for the case of a charged particle moving on a sphere in the background of a Dirac magnetic monopole as (see, e.g. [20] for details)

$$L = 2 D_t \bar{z} \cdot D_t z + \frac{i}{2} [\bar{\psi} \cdot D_t \psi - D_t \bar{\psi} \cdot \psi] - 2 g a, \quad (1)$$

where  $D_t z = (\partial_t - i a) z \equiv (\dot{z} - i a z)$ ,  $D_t \bar{z} = (\partial_t + i a) \bar{z} \equiv \dot{\bar{z}} + i a \bar{z}$ ,  $D_t \psi = (\partial_t - i a) \psi \equiv \dot{\psi} - i a \psi$ ,  $D_t \bar{\psi} = (\partial_t + i a) \bar{\psi} \equiv \dot{\bar{\psi}} + i a \bar{\psi}$  are the  $U(1)$  covariant derivatives under the  $CP^{(1)}$  model approach with the real “gauge” variable  $a$  and  $\partial_t = d/dt$  is the derivative w.r.t. the evolution parameter  $t$ . Here the electric charge  $e$  of the particle (with mass  $m = 1$ ) is taken to be  $e = -1$  and the magnetic charge on the monopole is denoted by  $g$ .

We concentrate on the *free* case of the above Lagrangian where  $a = 0$ . This leads to the following [20]

$$L_0 = 2 \dot{\bar{z}} \cdot \dot{z} + \frac{i}{2} (\bar{\psi} \cdot \dot{\psi} - \dot{\bar{\psi}} \cdot \psi), \quad (2)$$

where  $\bar{z} \cdot z = |z_1|^2 + |z_2|^2$  because we have taken  $\bar{z} = (\bar{z}_1 \ \bar{z}_2)$  and  $z = (z_1 \ z_2)^T$  as complex variables. Similar is the case with  $\bar{\psi} \cdot \psi = \bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2$  because  $\bar{\psi}$  and  $\psi$  are *independent* fermionic variables with  $\psi \cdot \psi = 0$  and  $\bar{\psi} \cdot \bar{\psi} = 0$  where  $\psi \cdot \psi \equiv \psi^T \cdot \psi = \psi_1^2 + \psi_2^2 = 0$ , etc.

The continuous and nilpotent ( $s_1^2 = s_2^2 = 0$ )  $\mathcal{N} = 2$  SUSY symmetry transformations of the above free Lagrangian  $L_0$  are as follows:

$$\begin{aligned} s_1 z &= \frac{\psi}{\sqrt{2}}, & s_1 \psi &= 0, & s_1 \bar{\psi} &= \frac{2i \dot{\bar{z}}}{\sqrt{2}}, & s_1 \bar{z} &= 0, \\ s_2 \bar{z} &= \frac{\bar{\psi}}{\sqrt{2}}, & s_2 \bar{\psi} &= 0, & s_2 \psi &= \frac{2i \dot{z}}{\sqrt{2}}, & s_2 z &= 0, \end{aligned} \quad (3)$$

because the Lagrangian  $L_0$  transforms to

$$s_1 L_0 = \frac{d}{dt} \left( \frac{\dot{\bar{z}} \cdot \psi}{\sqrt{2}} \right), \quad s_2 L_0 = \frac{d}{dt} \left( \frac{\bar{\psi} \cdot \dot{z}}{\sqrt{2}} \right). \quad (4)$$

As a consequence, the action integral  $S = \int dt L_0$  remains invariant. It is easy to check that the generators of the above transformations are the conserved charges:

$$Q = \frac{2 \dot{\bar{z}} \cdot \psi}{\sqrt{2}} \equiv \frac{\Pi_z \cdot \psi}{\sqrt{2}}, \quad \bar{Q} = \frac{2 \bar{\psi} \cdot \dot{z}}{\sqrt{2}} \equiv \frac{\bar{\psi} \cdot \Pi_{\bar{z}}}{\sqrt{2}}, \quad (5)$$

where the canonical conjugate momenta  $\Pi_z$  and  $\Pi_{\bar{z}}$  are w.r.t. variables  $z$  and  $\bar{z}$ . Similarly, the conjugate momenta w.r.t.  $\psi$  and  $\bar{\psi}$  in our theory are:  $\Pi_\psi = -(i/2) \bar{\psi}$  and  $\Pi_{\bar{\psi}} = -(i/2) \psi$  where the convention of the left-derivative w.r.t. the fermionic variables  $\psi$  and  $\bar{\psi}$  has been adopted. The conserved charges  $Q$  and  $\bar{Q}$  are the generators for  $s_1$  and  $s_2$  as can be seen from the following relationships:

$$s_r \Phi = \pm i [\Phi, Q_r]_{\pm} \quad r = 1, 2 \quad (Q_1 = Q, Q_2 = \bar{Q}), \quad (6)$$

where  $\Phi = z, \bar{z}, \psi, \bar{\psi}$  is the generic variable of our present theory and subscript  $(\pm)$  on the square bracket corresponds to the (anti)commutator for the generic variable  $\Phi$  being (fermionic) bosonic in nature.

The anticommutator of  $s_1$  and  $s_2$  (i.e.  $s_\omega = \{s_1, s_2\}$ ) generates a bosonic symmetry in the theory, namely;

$$s_\omega z = \dot{z}, \quad s_\omega \bar{z} = \dot{\bar{z}}, \quad s_\omega \psi = \dot{\psi}, \quad s_\omega \bar{\psi} = \dot{\bar{\psi}}, \quad (7)$$

modulo a factor of  $i$ . It is obvious that the generator of this time-translation is nothing but the Hamiltonian of our present free  $\mathcal{N} = 2$  SUSY theory. The explicit expression for the Hamiltonian (of our free  $\mathcal{N} = 2$  SUSY system) is

$$H = 2 \dot{\bar{z}} \cdot \dot{z} \equiv \frac{\Pi_z \cdot \Pi_{\bar{z}}}{2}, \quad (8)$$

where there is no potential (i.e. interaction) term.

### 3 Novel discrete symmetries

Under the following discrete transformations\*:

$$\begin{aligned} z &\rightarrow -\bar{z}, & \bar{z} &\rightarrow -z, & \psi &\rightarrow -\bar{\psi}, & \bar{\psi} &\rightarrow +\psi, & t &\rightarrow -t, \\ z &\rightarrow \pm i \bar{z}, & \bar{z} &\rightarrow \mp i z, & \psi &\rightarrow \pm i \bar{\psi}, & \bar{\psi} &\rightarrow \pm i \psi, & t &\rightarrow -t, \end{aligned} \quad (9)$$

the Lagrangian  $L_0$  remains invariant. We note that there is a time-reversal (i.e.  $t \rightarrow -t$ ) symmetry in the theory which implies, e.g.,  $z(t) \rightarrow z(-t) = \pm i \bar{z}^T$ , etc., in the latter transformations of (9). The above set of discrete symmetry transformations are the *novel* useful symmetries because they establish a set of connections between the nilpotent  $\mathcal{N} = 2$  symmetry transformations  $s_1$  and  $s_2$  as:

$$s_2 \Phi = \pm * s_1 * \Phi, \quad s_1 \Phi = \mp * s_2 * \Phi, \quad (10)$$

where  $(*)$  is nothing but the novel discrete symmetry transformations (9) for the generic variable  $\Phi = z, \bar{z}, \psi, \bar{\psi}$ .

For the duality-invariant theories [22], the  $(\pm)$  signs in (10) are governed by two successive  $(*)$  operations on the generic variable  $\Phi$ , namely;

$$* (* \Phi) = \pm \Phi, \quad \Phi = z, \bar{z}, \psi, \bar{\psi}. \quad (11)$$

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\* In these discrete symmetry transformations, we have suppressed the explicit notations for the transpose operations on the dynamical variables  $z, \bar{z}, \psi, \bar{\psi}$  of our SUSY quantum mechanical theory.

In our case, it can be explicitly checked that

$$\begin{aligned} * (* \Phi_1) &= + \Phi_1, & \Phi_1 &= z, \bar{z}, \\ * (* \Phi_2) &= - \Phi_2, & \Phi_2 &= \psi, \bar{\psi}. \end{aligned} \quad (12)$$

The  $(\pm)$  signs in  $s_2\Phi = \pm * s_1 * \Phi$  (cf. (10)) are the same as those given in (11). However, it is the reverse signature that is true for  $s_1\Phi = \mp * s_2 * \Phi$  *vis-à-vis* Eq. (11).

It is interesting to note that, under the following set of discrete transformations:

$$\begin{aligned} t &\rightarrow t, & z &\rightarrow \pm i \bar{z}, & \bar{z} &\rightarrow \mp i z, & \psi &\rightarrow \pm i \bar{\psi}, & \bar{\psi} &\rightarrow \mp i \psi, \\ t &\rightarrow t, & z &\rightarrow \pm i \bar{z}, & \bar{z} &\rightarrow \mp i z, & \psi &\rightarrow \bar{\psi}, & \bar{\psi} &\rightarrow \psi, \\ t &\rightarrow t, & z &\rightarrow \bar{z}, & \bar{z} &\rightarrow z, & \psi &\rightarrow \bar{\psi}, & \bar{\psi} &\rightarrow \psi, \end{aligned} \quad (13)$$

the Lagrangian remains invariant. We note that there is *no* time-reversal symmetry in the above transformations (i.e.  $t \rightarrow t$ ). Further, it is elementary to note that  $*(*\Phi) = +\Phi$  for the generic variable  $\Phi = z, \bar{z}, \psi, \bar{\psi}$ . However, it can be explicitly checked that the relationships:

$$s_2\Phi = + * s_1 * \Phi, \quad s_1\Phi \neq - * s_2 * \Phi, \quad (14)$$

are *true* for the top and bottom symmetry transformations in (13). Thus, we note that the reciprocal relationship (i.e.  $s_1\Phi = - * s_2 * \Phi$ ) is *not* satisfied. It is remarkable to note that *even* the first relationship of (14) is *not* satisfied by the discrete transformations pointed out in the middle of (13). Thus, according to the rules laid down in [22] for developing the duality-invariant theories, the discrete transformations (13) are *not* useful to us.

## 4 Algebraic structures

As we have seen, there are *three* continuous symmetries (i.e.  $s_1, s_2, s_\omega$ ) in the theory. The continuous symmetries, in their operator form, satisfy

$$\begin{aligned} s_1^2 &= 0, & s_2^2 &= 0, & \{s_1, s_2\} &= s_\omega = (s_1 + s_2)^2, \\ [s_\omega, s_1] &= 0, & [s_\omega, s_2] &= 0, & \{s_1, s_2\} &\neq 0, \end{aligned} \quad (15)$$

when they operate on the generic field  $\Phi = z, \bar{z}, \psi, \bar{\psi}$  of the theory. The above algebra is reminiscent of the algebra obeyed by the de Rham cohomological operators of differential geometry (see, e.g. [23,24]), namely;

$$\begin{aligned} d^2 &= 0, & \delta^2 &= 0, & \{d, \delta\} &= \Delta = (d + \delta)^2, \\ [\Delta, d] &= 0, & [\Delta, \delta] &= 0, & \{d, \delta\} &\neq 0, \end{aligned} \quad (16)$$

where  $d$  (with  $d^2 = 0$ ) is the exterior derivative  $\delta$  (with  $\delta^2 = 0$ ) is the (co-)exterior derivative and  $\Delta = (d + \delta)^2$  is the Laplacian operator.

In an exactly similar fashion, it will be noted that the conserved charges  $(Q, \bar{Q}, H)$  also obey the following algebra

$$Q^2 = \bar{Q}^2 = 0, \quad \{Q, \bar{Q}\} = H, \quad [H, Q] = 0, \quad [H, \bar{Q}] = 0, \quad H = (Q + \bar{Q})^2, \quad (17)$$

in the case of our present theory. The latter two entries (i.e.  $[H, Q] = 0$ ,  $[H, \bar{Q}] = 0$ ) are nothing but the conservation law for the charges  $Q$  and  $\bar{Q}$  (which can be checked easily by *either* using directly the Euler-Lagrange equations of motion *or* the basic brackets  $[z, \Pi_z] = [\bar{z}, \Pi_{\bar{z}}] = i$ ,  $\{\psi, \bar{\psi}\} = +1$ ). The algebra in (17) is one of simplest forms [25] of the  $\mathcal{N} = 2$  SUSY algebra  $sl(1/1)$ .

A close look at (15), (16) and (17) demonstrates that, at the algebraic level, all these equations are equivalent. However, we have still not been able to provide the physical realization of the very important relationship  $\delta = \pm * d *$  that exists between the (co-) exterior derivatives  $(\delta)d$  of differential geometry. In this connection, it is pertinent to point out that the relationship (10) (cf. Sec. 3) provides the physical realization of  $\delta = \pm * d *$  and  $d = \mp * \delta *$  in terms of the innate continuous and discrete symmetries of our present theory.

## 5 Towards cohomological aspects

We have noted in the previous section that  $H$  is the Casimir operator of the algebra (17). Thus, it is clear that  $HQ = QH$  and  $H\bar{Q} = \bar{Q}H$  imply that  $QH^{-1} = H^{-1}Q$  and  $\bar{Q}H^{-1} = H^{-1}\bar{Q}$ . Using (17), it can be seen that

$$\begin{aligned} \left[ \frac{Q\bar{Q}}{H}, Q \right] &= +Q, & \left[ \frac{Q\bar{Q}}{H}, \bar{Q} \right] &= -\bar{Q}, \\ \left[ \frac{\bar{Q}Q}{H}, \bar{Q} \right] &= +\bar{Q}, & \left[ \frac{\bar{Q}Q}{H}, Q \right] &= -Q. \end{aligned} \quad (18)$$

As a consequence of the above equation, it is evident that if we define the eigenvalue equation:  $(Q\bar{Q}/H) |\psi\rangle_q = q |\psi\rangle_q$  for a state  $|\psi\rangle_q$  in the quantum Hilbert space of states, then, we have the validity of the following

$$\begin{aligned} \left( \frac{Q\bar{Q}}{H} \right) Q |\psi\rangle_q &= (q+1) Q |\psi\rangle_q, \\ \left( \frac{Q\bar{Q}}{H} \right) \bar{Q} |\psi\rangle_q &= (q-1) \bar{Q} |\psi\rangle_q, \\ \left( \frac{Q\bar{Q}}{H} \right) H |\psi\rangle_q &= q H |\psi\rangle_q, \end{aligned} \quad (19)$$

where  $q$  is the eigenvalue of state  $|\psi\rangle_q$  w.r.t. the hermitian operator  $(Q\bar{Q}/H)$ . This observation implies that  $q$  is a real number. A close look at (19) demonstrates that  $Q |\psi\rangle_q$ ,  $\bar{Q} |\psi\rangle_q$  and  $H |\psi\rangle_q$  have the eigenvalues  $(q+1)$ ,  $(q-1)$  and  $q$ , respectively, w.r.t. the operator  $(Q\bar{Q}/H)$  which is a physical operator.

The above observation in (19) establishes a connection between the set of conserved charges  $(Q, \bar{Q}, H)$  and the set of de Rham cohomological operators  $(d, \delta, \Delta)$  because, as we know, the operation of  $d$  on a differential form of degree  $q$ , raises the degree of the form by one (i.e.  $d f^{(q)} \sim f^{(q+1)}$ ). On the contrary, the action of  $\delta$ , on a  $q$ -form, lowers the degree of form by one (i.e.  $\delta f^{(q)} \sim f^{(q-1)}$ ). Finally, we note that  $\Delta f^{(q)} \sim f^{(q)}$  which shows that

the degree of a form remains intact when it is operated upon by the Laplacian operator  $\Delta$  of differential geometry.

In our present free  $\mathcal{N} = 2$  SUSY theory, there is yet another physical realization of the cohomological operators because we note that if we take a state  $|\chi\rangle_p$  which has an eigenvalue  $p$  corresponding to  $(\bar{Q}Q/H)$ , namely;

$$\left(\frac{\bar{Q}Q}{H}\right) |\chi\rangle_p = p |\chi\rangle_p, \quad (20)$$

(where  $p$  is a real number), then, we have the following

$$\begin{aligned} \left(\frac{\bar{Q}Q}{H}\right) \bar{Q} |\chi\rangle_p &= (p+1) \bar{Q} |\chi\rangle_p, \\ \left(\frac{\bar{Q}Q}{H}\right) Q |\chi\rangle_p &= (p-1) Q |\chi\rangle_p, \\ \left(\frac{\bar{Q}Q}{H}\right) H |\chi\rangle_p &= p H |\chi\rangle_p, \end{aligned} \quad (21)$$

which demonstrates that the states  $\bar{Q} |\chi\rangle_p$ ,  $Q |\chi\rangle_p$  and  $H |\chi\rangle_p$  have the eigenvalues  $(p+1)$ ,  $(p-1)$  and  $p$ , respectively. This crucial observation is *also* identical to the consequences that emerge after operation of the set  $(d, \delta, \Delta)$  on a differential form of degree  $p$ . Thus, we have the following mapping:

$$(\bar{Q}, Q, H) \Longleftrightarrow (d, \delta, \Delta). \quad (22)$$

We conclude that, for our  $\mathcal{N} = 2$  SUSY theory, there are *two* physical realizations of  $(d, \delta, \Delta)$  in the language of conserved charges and their eigenvalues. If the degree of a given differential form is identified with the eigenvalue of a state in the total quantum Hilbert space of states (w.r.t. a specific hermitian operator), then, the operations of  $(d, \delta, \Delta)$  on the above form exactly match with the operations of conserved charges on the specifically chosen quantum state of the theory in the Hilbert space.

## 6 $\mathcal{N} = 2$ SUSY symmetries: supervariable approach

We can derive the nilpotent ( $s_1^2 = 0, s_2^2 = 0$ ) symmetries  $s_1$  and  $s_2$  by using the supervariable approach [12] where the SUSY invariant restrictions (SUSYIRs) play very important role. For the derivation of  $s_1$  (cf. (3)), first of all, we generalize the dynamical variables  $(z(t), \bar{z}(t), \psi(t), \bar{\psi}(t))$  to their counterparts supervariables on the chiral supermanifold  $(Z(t, \theta), \bar{Z}(t, \theta), \Psi(t, \theta), \bar{\Psi}(t, \theta))$  with the following expansions along the Grassmannian  $\theta$ -direction (see, e.g. [12]):

$$\begin{aligned} z(t) &\longrightarrow Z(t, \theta) = z(t) + \theta f_1(t), \\ \bar{z}(t) &\longrightarrow \bar{Z}(t, \theta) = \bar{z}(t) + \theta f_2(t), \\ \psi(t) &\longrightarrow \Psi(t, \theta) = \psi(t) + i\theta b_1(t), \\ \bar{\psi}(t) &\longrightarrow \bar{\Psi}(t, \theta) = \bar{\psi}(t) + i\theta b_2(t), \end{aligned} \quad (23)$$

where the (1, 1)-dimensional chiral supermanifold is parametrized by  $(t, \theta)$  and, as is evident, the secondary variables  $(f_1, f_2)$  and  $(b_1, b_2)$  are fermionic and bosonic in nature, respectively.

As has been pointed out in [12], the SUSYIRs require that the SUSY invariant quantities should be independent of the “soul” coordinates  $\theta$  and  $\bar{\theta}$  (with  $\theta^2 = \bar{\theta}^2 = 0$ ,  $\theta\bar{\theta} + \bar{\theta}\theta$ ). For instance, it is clear (from (3)) that  $s_1\bar{z} = 0$ ,  $s_1\psi = 0$ , which implies that we have the following SUSYIRs:

$$\bar{Z}(t, \theta) = \bar{z}(t), \quad \Psi(t, \theta) = \psi(t) \implies f_2 = 0, \quad b_1 = 0. \quad (24)$$

As a consequence, we have the  $\theta$ -independence of the chiral supervariables  $\bar{Z}(t, \theta)$  and  $\Psi(t, \theta)$ . Now we note that  $s_1(z^T \cdot \psi) = 0$  and  $s_1(\dot{\bar{z}} \cdot z + \frac{i}{2}\bar{\psi} \cdot \psi) = 0$ . Thus, we have the following SUSYIRs in our theory, namely;

$$\begin{aligned} Z^T(t, \theta) \cdot \Psi(t, \theta) &= z^T(t) \cdot \psi(t), \\ \dot{\bar{Z}}(t, \theta) \cdot Z(t, \theta) + \frac{i}{2}\bar{\Psi}(t, \theta) \cdot \Psi(t, \theta) &= \dot{\bar{z}}(t) \cdot z(t) + \frac{i}{2}\bar{\psi}(t) \cdot \psi(t), \end{aligned} \quad (25)$$

where  $z^T \cdot \psi = z_1\psi_1 + z_2\psi_2$ . Taking the help of (24), it is clear that  $f_1(t) \propto \psi(t)$  because of the top restriction in (25). Choosing  $f_1(t) = (\psi(t)/\sqrt{2})$ , we obtain  $b_2(t) = (2\dot{\bar{z}}(t)/\sqrt{2})$  from the bottom restriction of (25). Plugging in these values in the expansions (23), we obtain

$$\begin{aligned} Z^{(1)}(t, \theta) &= z(t) + \theta \left( \frac{\psi}{\sqrt{2}} \right) \equiv z(t) + \theta(s_1 z), \\ \bar{Z}^{(1)}(t, \theta) &= \bar{z}(t) + \bar{\theta}(0) \equiv \bar{z}(t) + \theta(s_1 \bar{z}), \\ \Psi^{(1)}(t, \theta) &= \psi(t) + \theta(0) \equiv \psi(t) + \theta(s_1 \psi), \\ \bar{\Psi}^{(1)}(t, \theta) &= \bar{\psi}(t) + \theta \left( \frac{2i\dot{\bar{z}}}{\sqrt{2}} \right) \equiv \bar{\psi}(t) + \theta(s_1 \bar{\psi}), \end{aligned} \quad (26)$$

where the superscript (1) on the supervariables stands for the expansions of the chiral supervariables after application of the SUSYIRs (25).

A close look at (26) demonstrates that we have already obtained the transformations  $s_1$  (cf. (3)). Furthermore, we observe that the following mapping is true:

$$\frac{\partial}{\partial \theta} \left( \Omega^{(1)}(t, \theta) \right) = s_1 \Omega(t), \quad (27)$$

which establishes the connection between the translational generator  $\partial_\theta$  on the chiral (1, 1)-dimensional supermanifold and the SUSY transformations  $s_1$ . In (27),  $\Omega^{(1)}(t, \theta)$  is the generic supervariable obtained in (26) and  $\Omega(t) = z, \bar{z}, \psi, \bar{\psi}$  denotes the generic dynamical variable of the Lagrangian  $L_0$  (cf. (2)).

For the derivation of nilpotent ( $s_2^2 = 0$ ) transformations  $s_2$ , first of all, we generalize the dynamical variables  $(z(t), \bar{z}(t), \psi(t), \bar{\psi}(t))$  to their counterparts supervariables  $(Z(t, \bar{\theta}), \bar{Z}(t, \bar{\theta}), \Psi(t, \bar{\theta}), \bar{\Psi}(t, \bar{\theta}))$  on the anti-chiral supermanifold with the following general



expansions

$$\begin{aligned}
z(t) &\longrightarrow Z(t, \bar{\theta}) = z(t) + \bar{\theta} f_3(t), \\
\bar{z}(t) &\longrightarrow \bar{Z}(t, \bar{\theta}) = \bar{z}(t) + \bar{\theta} f_4(t), \\
\psi(t) &\longrightarrow \Psi(t, \bar{\theta}) = \psi(t) + i \bar{\theta} b_3(t), \\
\bar{\psi}(t) &\longrightarrow \bar{\Psi}(t, \bar{\theta}) = \bar{\psi}(t) + i \bar{\theta} b_4(t),
\end{aligned} \tag{28}$$

where the secondary variables  $(f_3, f_4)$  and  $(b_3, b_4)$  are fermionic and bosonic, respectively. We note that  $s_2 z = 0, s_2 \bar{\psi} = 0$ . Thus, we have the following SUSYIRs:

$$Z(t, \bar{\theta}) = z(t), \quad \bar{\Psi}(t, \bar{\theta}) = \bar{\psi}(t) \implies f_3 = 0, \quad b_4 = 0. \tag{29}$$

To determine the other secondary variables  $(f_4(t), b_3(t))$ , we have the following SUSYIRs:

$$\begin{aligned}
\bar{Z}(t, \bar{\theta}) \cdot \bar{\Psi}^T(t, \bar{\theta}) &= \bar{z}(t) \cdot \bar{\psi}^T(t), \\
\bar{Z}(t, \bar{\theta}) \cdot \dot{\bar{Z}}(t, \bar{\theta}) - \frac{i}{2} \bar{\Psi}(t, \bar{\theta}) \cdot \Psi(t, \bar{\theta}) &= \bar{z}(t) \cdot \dot{\bar{z}}(t) - \frac{i}{2} \bar{\psi}(t) \cdot \psi(t),
\end{aligned} \tag{30}$$

because we note that the above expressions remain invariant  $s_2(\bar{z} \cdot \bar{\psi}^T) = 0, s_2(\bar{z} \cdot \dot{\bar{z}} - \frac{i}{2} \bar{\psi} \cdot \psi) = 0$  under  $s_2$ . Using inputs from (29), we obtain the following:

$$f_4(t) = \frac{\bar{\psi}(t)}{\sqrt{2}}, \quad b_3(t) = \frac{2 \dot{\bar{z}}(t)}{\sqrt{2}}. \tag{31}$$

Thus, the expansions (28) reduce to

$$\begin{aligned}
Z^{(2)}(t, \bar{\theta}) &= z(t) + \bar{\theta}(0) \equiv z(t) + \bar{\theta}(s_2 z), \\
\bar{Z}^{(2)}(t, \bar{\theta}) &= \bar{z}(t) + \bar{\theta} \left( \frac{\bar{\psi}}{\sqrt{2}} \right) \equiv \bar{z}(t) + \bar{\theta}(s_2 \bar{z}), \\
\Psi^{(2)}(t, \bar{\theta}) &= \psi(t) + \bar{\theta} \left( \frac{2i \dot{\bar{z}}}{\sqrt{2}} \right) \equiv \psi(t) + \bar{\theta}(s_2 \psi), \\
\bar{\Psi}^{(2)}(t, \bar{\theta}) &= \bar{\psi}(t) + \bar{\theta}(0) \equiv \bar{\psi}(t) + \bar{\theta}(s_2 \bar{\psi}),
\end{aligned} \tag{32}$$

where the superscript (2) denotes the expansions of the supervariables after the application of the SUSYIRs (29) and (30). A careful observation of (32) demonstrates that we have already derived the SUSY transformations  $s_2$  (cf. (3)). It is worth mentioning that the analogue of (27) can be defined for the SUSY transformations  $s_2$  as well.

To provide the geometrical meaning to the nilpotency of the conserved charges  $Q$  and

$\bar{Q}$ , we note the following

$$\begin{aligned}
Q &= \frac{\partial}{\partial \theta} \left[ 2 \dot{Z}^{(1)}(t, \theta) \cdot Z^{(1)}(t, \theta) \right] \equiv \frac{\partial}{\partial \theta} \left[ 2 \dot{z}(t) \cdot Z^{(1)}(t, \theta) \right], \\
&= \int d\theta \left[ 2 \dot{Z}^{(1)}(t, \theta) \cdot Z^{(1)}(t, \theta) \right] \equiv \int d\theta \left[ 2 \dot{z}(t) \cdot Z^{(1)}(t, \theta) \right], \\
Q &= \frac{\partial}{\partial \theta} \left[ -i \bar{\Psi}^{(1)}(t, \theta) \cdot \Psi^{(1)}(t, \theta) \right] \equiv \frac{\partial}{\partial \theta} \left[ -i \bar{\Psi}^{(1)}(t, \theta) \cdot \psi(t) \right], \\
&= \int d\theta \left[ -i \bar{\Psi}^{(1)}(t, \theta) \cdot \Psi^{(1)}(t, \theta) \right] \equiv \int d\theta \left[ -i \bar{\Psi}^{(1)}(t, \theta) \cdot \psi(t) \right], \\
\bar{Q} &= \frac{\partial}{\partial \bar{\theta}} \left[ 2 \bar{Z}^{(2)}(t, \bar{\theta}) \cdot \dot{Z}^{(2)}(t, \bar{\theta}) \right] \equiv \frac{\partial}{\partial \bar{\theta}} \left[ 2 \bar{Z}^{(2)}(t, \bar{\theta}) \cdot \dot{z}(t) \right], \\
&= \int d\bar{\theta} \left[ 2 \bar{Z}^{(2)}(t, \bar{\theta}) \cdot \dot{Z}^{(2)}(t, \bar{\theta}) \right] \equiv \int d\bar{\theta} \left[ 2 \bar{Z}^{(2)}(t, \bar{\theta}) \cdot \dot{z}(t) \right], \\
\bar{Q} &= \frac{\partial}{\partial \bar{\theta}} \left[ +i \bar{\Psi}^{(2)}(t, \bar{\theta}) \cdot \Psi^{(2)}(t, \bar{\theta}) \right] \equiv \frac{\partial}{\partial \bar{\theta}} \left[ +i \bar{\psi}(t) \cdot \Psi^{(2)}(t, \bar{\theta}) \right], \\
&= \int d\bar{\theta} \left[ +i \bar{\Psi}^{(2)}(t, \bar{\theta}) \cdot \Psi^{(2)}(t, \bar{\theta}) \right] \equiv \int d\bar{\theta} \left[ +i \bar{\psi}(t) \cdot \Psi^{(2)}(t, \bar{\theta}) \right]. \tag{33}
\end{aligned}$$

Thus, there are two different ways to express  $Q$  and  $\bar{Q}$  in the language of supervariables and Grassmannian derivatives. The nilpotency of  $\partial_\theta$  and  $\partial_{\bar{\theta}}$  (i.e.  $\partial_\theta^2 = 0$ ,  $\partial_{\bar{\theta}}^2 = 0$ ) implies that  $\partial_\theta Q = 0$ ,  $\partial_{\bar{\theta}} \bar{Q} = 0$ . The latter imply expressions, in the language of SUSY transformations  $s_1$  and  $s_2$  (and their generators) as:  $s_1 Q = i \{Q, Q\} = 0$  and  $s_2 \bar{Q} = i \{\bar{Q}, \bar{Q}\} = 0$  which prove the nilpotency ( $Q^2 = \bar{Q}^2 = 0$ ) of the SUSY charges  $Q$  and  $\bar{Q}$ . Furthermore, when we express the above expressions for  $Q$  and  $\bar{Q}$  (cf. (33)) in terms of the ordinary SUSY symmetries and dynamical variables, we observe that

$$Q = s_1 \left( 2 \dot{z} \cdot z \right) \equiv s_1 \left( -i \bar{\psi} \cdot \psi \right), \quad \bar{Q} = s_2 \left( 2 \bar{z} \cdot \dot{z} \right) \equiv s_2 \left( +i \bar{\psi} \cdot \psi \right). \tag{34}$$

The above expressions also prove the nilpotency of the conserved charges  $Q$  and  $\bar{Q}$  in a straightforward manner because  $s_1^2 = 0$ ,  $s_2^2 = 0$ .

We can also capture the invariance of the Lagrangian  $L_0$  (cf. (2)) in terms of the supervariables obtained after SUSYIRs. For instance, it can be checked explicitly that the following generalizations of (2), namely;

$$\begin{aligned}
L_0 &\implies \tilde{L}_0^{(ac)} = 2 \dot{Z}^{(2)} \cdot \dot{Z}^{(2)} + \frac{i}{2} \left[ \bar{\Psi}^{(2)} \cdot \dot{\Psi}^{(2)} - \dot{\bar{\Psi}}^{(2)} \cdot \Psi^{(2)} \right], \\
L_0 &\implies \tilde{L}_0^{(c)} = 2 \dot{Z}^{(1)} \cdot \dot{Z}^{(1)} + \frac{i}{2} \left[ \bar{\Psi}^{(1)} \cdot \dot{\Psi}^{(1)} - \dot{\bar{\Psi}}^{(1)} \cdot \Psi^{(1)} \right], \tag{35}
\end{aligned}$$

(where the superscripts  $(c)$  and  $(ac)$  denote the chiral and anti-chiral nature of the Lagrangians  $\tilde{L}_0^{(c)}$  and  $\tilde{L}_0^{(ac)}$ , respectively) lead to one of the key observations that

$$\begin{aligned}
\frac{\partial}{\partial \theta} \left[ \tilde{L}_0^{(c)} \right] &= s_1 L_0 \equiv \frac{d}{dt} \left( \frac{\dot{z} \cdot \psi}{\sqrt{2}} \right), \\
\frac{\partial}{\partial \bar{\theta}} \left[ \tilde{L}_0^{(ac)} \right] &= s_2 L_0 \equiv \frac{d}{dt} \left( \frac{\bar{\psi} \cdot \dot{z}}{\sqrt{2}} \right). \tag{36}
\end{aligned}$$

Geometrically, the invariance (cf. (4)) of the free Lagrangian is encoded in the above expressions (36). It states that  $\tilde{L}_0^{(c)}$  and  $\tilde{L}_0^{(ac)}$  are the sum of composite supervariables constructed from (26) and (32) such that their translations along the  $\theta$  and  $\bar{\theta}$ -directions of the (1, 1)-dimensional chiral and anti-chiral supermanifolds, respectively, produce the ordinary time-derivatives in the ordinary 1D space thereby leading to symmetry invariance.

## 7 Conclusions

Our present endeavor is our first modest step towards our main goal of proving  $\mathcal{N} = 2$  and  $\mathcal{N} = 4$  interacting theories (of a charged particle constrained to move on a sphere in the background of a Dirac magnetic monopole) as tractable SUSY models for the Hodge theory. To achieve the above mentioned central goals, first of all, we have considered the *free*  $\mathcal{N} = 2$  version of the above interacting systems and established that it is a model for the Hodge theory.

In our present investigation, we have shown the physical realizations of the de Rham cohomological operators in the language of symmetries (and conserved charges). We have also derived the  $\mathcal{N} = 2$  nilpotent SUSY transformations by exploiting the supervariable approach [12] and provided geometrical meanings to them. In fact, as it turns out, the nilpotent  $\mathcal{N} = 2$  SUSY transformations ( $s_1$  and  $s_2$ ) are nothing but the translational generators ( $\partial_\theta$  and  $\partial_{\bar{\theta}}$ ) along the  $\theta$  and  $\bar{\theta}$ -directions of the chiral and anti-chiral supermanifolds on which the (0 + 1)-dimensional dynamical variables are generalized as supervariables. The nilpotency of the transformations  $s_1$  and  $s_2$  are also encoded in such properties associated with  $\partial_\theta$  and  $\partial_{\bar{\theta}}$ .

One of our immediate goals is to prove that the interacting  $\mathcal{N} = 2$  SUSY quantum mechanical model of a charged particle, constrained to move on a sphere in the background of a Dirac magnetic monopole [20], is a tractable SUSY model for the Hodge theory. The most interesting future endeavor for us is to find out the physical realizations of the cohomological operators in the case of  $\mathcal{N} = 4$  SUSY quantum mechanical model [21] in the language of symmetry properties and conserved charges.

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